Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 14 June 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/3C**

Further Mathematics

Advanced

PAPER 3C: Further Mechanics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Unless otherwise indicated, whenever a value of g is required, take $g = 9.8$ m s^{-2} and give your answer to either 2 significant figures or 3 significant figures.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

1. A particle P of mass 2 kg is moving with velocity $(-4\mathbf{i} + 3\mathbf{j})$ m s^{-1} when it receives an impulse $(-6\mathbf{i} + 42\mathbf{j})$ N s.

(a) Find the speed of P immediately after receiving the impulse.

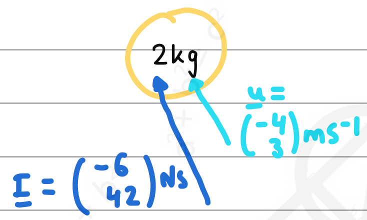
(4)

The angle through which the direction of motion of P has been deflected by the impulse is $\alpha°$

(b) Find the value of α

(2)

(a) realising that we're dealing with vector impulse and momentum - hence illustrating the particle's motion diagrammatically - label the velocity before and the Impulse



in order to get the speed after, first need to find the velocity after the particle receives the Impulse - hence subbing above information into the vector version of the Impulse-momentum formula:

formula: $\underline{I} = m(\underline{v} - \underline{u})$

$$\begin{pmatrix}-6\\42\end{pmatrix} = 2\left(\underline{v} - \begin{pmatrix}-4\\3\end{pmatrix}\right)$$

expand RHS

$$\begin{pmatrix}-6\\42\end{pmatrix} = 2\underline{v} - \begin{pmatrix}-8\\6\end{pmatrix}$$

and rearrange and solve for $\underline{v}$

$$2\underline{v} = \begin{pmatrix}-6\\42\end{pmatrix} + \begin{pmatrix}-8\\6\end{pmatrix}$$

$$2\underline{v} = \begin{pmatrix}-14\\48\end{pmatrix}$$

$\div 2$ $\qquad$ $\div 2$

$$\underline{v} = \begin{pmatrix}-7\\24\end{pmatrix} \text{ ms}^{-1}$$

but asked for speed, not velocity, hence Pythagorise $\underline{v}$

$$|\underline{v}| = \sqrt{(-7)^2 + (24)^2}$$

$$= \sqrt{49 + 576}$$

$$= \sqrt{625} = \boxed{25 \text{ ms}^{-1}}$$

(b) now the question is asking us to find the angle of deflection - i.e the angle between $\underline{u}_P = \begin{pmatrix}-4\\3\end{pmatrix}$ and $\underline{v}_P = \begin{pmatrix}-7\\24\end{pmatrix}$ - two ways to do this:

given $\qquad$ from part (a)

Question 1 continued

WAY 1: using scalar product formula

formula: $\cos\alpha = \dfrac{a \cdot b}{|a||b|}$ ← scalar product

product of magnitudes

$$\cos\alpha = \frac{\begin{pmatrix}-4\\3\end{pmatrix}\cdot\begin{pmatrix}-7\\24\end{pmatrix}}{\sqrt{(-4)^2+(3)^2}\sqrt{(-7)^2+(24)^2}}$$

$$= \frac{28+72}{\sqrt{25}\sqrt{625}}$$

$$\cos\alpha = \frac{100}{125}$$

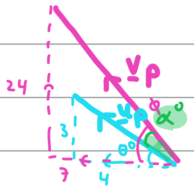
take inverse cos for α°

$$\alpha = \cos^{-1}\left(\frac{100}{125}\right)$$

$$= 36.869..$$

$$\boxed{= 37^\circ (2 \text{ s.f})}$$

WAY 2: using trig and triangles



see how angle α° is obtained from angle from blue triangle - call it θ° - subtracted from angle from pink triangle - call it 'ϕ'

$$\alpha = \phi - \theta$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{3}{4}\right)$$

$$= 73.7397... - 36.8698...$$

$$= 36.86939... = \boxed{37^\circ (\text{to } 2 \text{ s.f})}$$

(Total for Question 1 is 6 marks)

2. A car of mass 1000 kg moves in a straight line along a horizontal road at a constant speed U m s^{-1}. The resistance to the motion of the car is a constant force of magnitude 400 N.

The engine of the car is working at a constant rate of 16 kW.

(a) Find the value of U.

(3)

The car now pulls a trailer of mass 600 kg in a straight line along the road using a tow rope which is parallel to the direction of motion. The resistance to the motion of the car is again a constant force of magnitude 400 N. The resistance to the motion of the trailer is a constant force of magnitude 300 N.

The engine of the car is working at a constant rate of 16 kW.

The tow rope is modelled as being light and inextensible.

Using the model,

(b) find the tension in the tow rope at the instant when the speed of the car is $\frac{20}{3}$ m s^{-1}

(5)

(a) let's illustrate the above information on a detailed FORCE diagram

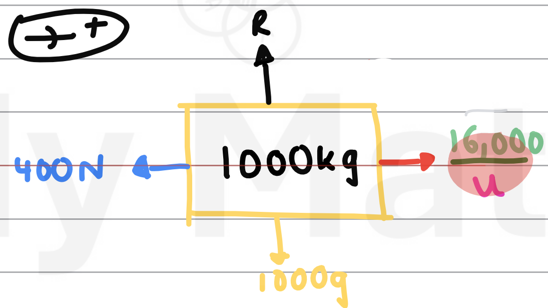
↳ label the resistance to motion, the weight and the power rearranged

formula: $P = Fv$ (Power in Watts, Force in Newtons, velocity in ms^{-1})

$F = \frac{P}{v}$

where $P = 16\text{kW} \xrightarrow[\text{watts } (\times 1000)]{\text{convert to}} 16{,}000\text{W}$

and $v = U$



NOTE: could've calculated this as a separate line of working but much better in exams to straight away write in the force from the formula for power rearranged

we know from Yr 2 Mechanics Chp 8 that if the car is moving at 'constant speed' this suggests it's in non-stationary equilibrium ∴ "forces left" = "forces right"

$R(\leftrightarrow)$: $\frac{16{,}000}{U} = 400$

$\times U \qquad \times U$

$16{,}000 = 400U$

$\div 400 \qquad \div 400$

$\Rightarrow \boxed{U = 40\text{ms}^{-1}}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 2 continued

(b) now the car is still moving on a straight, horizontal ground but it's towing a trailer

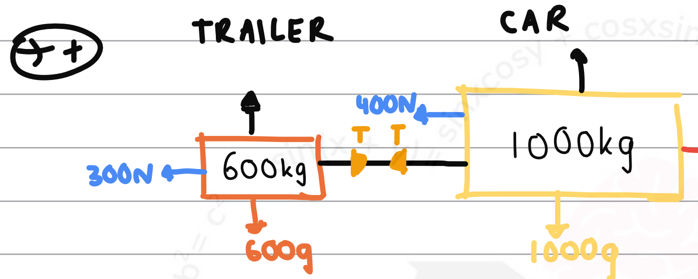
↳ illustrating this diagramatically, labelling resistance, tension in the towrope and the power rearranged:

formula: $P = Fv$

$F = \frac{P}{v}$

where $16kW \xrightarrow[\text{(}\times 1000\text{)}]{\text{convert to units}} 16{,}000$

and $v = 20/3$



$\frac{16{,}000}{20/3} = 2{,}400$

remembering from Yr 2 Mechanics Chp 8 that if consider a system as a whole, the tension in the tow rope will cancel out - even though this isn't immediately helpful to us to get the tension directly, it will help us get the acceleration we need to ultimately get the tension in the tow rope

↳ using Newton's Second Law on the system to get the acceleration:

formula: $\Sigma F_x = ma$

$R(\leftrightarrow): 2{,}400 - 400 - 300 = (1000 + 600)a$

expand and collect terms

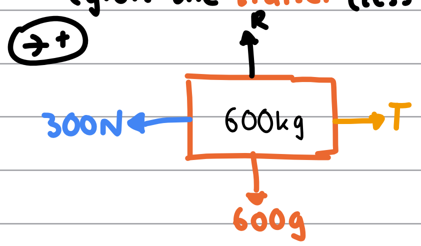
$1600a = 1700$

$\div 1700 \quad \div 1700$

$\Rightarrow a = \frac{16}{17} \text{ ms}^{-2}$

now that we know the value for 'a' we can use Newton's Second Law - but this time on any of the trailer or the car

eg. on the trailer (less time-consuming)



formula: $\Sigma F_x = ma$

$R(\leftrightarrow): T - 300 = 600\left(\frac{17}{16}\right)$

expand and rearrange for T

Turn over

P 7 2 7 9 8 A 0 5 2 8

Question 2 continued

$T = 300 + \frac{1275}{2}$

$= \frac{1875}{2} = 937.5\text{N}$

$\boxed{\therefore 938\text{ N } (3\text{s.f})}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 2 continued

(Total for Question 2 is 8 marks)

3. A particle P of mass $2m$ is moving in a straight line with speed $3u$ on a smooth horizontal plane. It collides directly with a particle Q of mass m that is moving on the plane with speed $2u$ in the opposite direction to P.

The coefficient of restitution between P and Q is e, where $e > \frac{4}{5}$

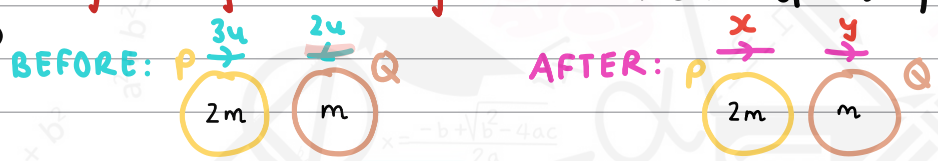
(a) Show that the speed of Q immediately after the collision is $\frac{(4+10e)u}{3}$ **(6)**

After the collision Q hits a smooth fixed vertical wall that is perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is f.

(b) Find, **in terms of** e, the set of values of f for which there will be a second collision between P and Q. **(4)**

(a) first part of the question is a typical 'elastic collisions in 1D' question - illustrating it diagrammatically - label the direction of motion, respective speeds etc.

$\rightarrow +$



following the normal procedure for these types of collision - notice both the velocities after are unknown, so can't just stop at PCLM - have to do NEL (Impact law) as well

...first, PCLM - i.e total momentum before equals total momentum after:

formula: $m_P u_P + m_Q u_Q = m_P v_P + m_Q v_Q$

sub into above

$2\cancel{m}(3u) + \cancel{m}(-2u) = 2\cancel{m}x + \cancel{m}y$

cancel m's and expand brackets

$2x + y = 6u - 2u$

$\Rightarrow 2x + y = 4u$ ①

...next, NEL:

formula: $e = \frac{\text{speed of separation}}{\text{speed of approach}}$

$e = \frac{v_Q - v_P}{u_P - u_Q}$

$e = \frac{y - (-x)}{3u - (-2u)}$

$\Rightarrow e = \frac{y - x}{5u}$

$\times 5u \qquad \times 5u$

P 7 2 7 9 8 A 0 8 2 8

Question 3 continued

$\Rightarrow \quad y - x = 5eu$ ②

need to solve ① and ② simultaneously – elim. 'x'

① + 2×②

$$2x + y = 4u$$
$$+ \; -2x + 2y = 10eu$$
$$3y = 4u + 10eu$$

factorise 'u' out and solve for 'y'

÷3 ÷3

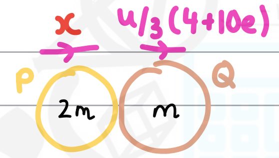
$$\boxed{y = \frac{u}{3}(4 + 10e)}$$

(b) now we're introduced to a second collision – this time one between particle Q and a vertical wall

↳ illustrating this diagrammatically:

→+

BEFORE SECOND COLLISION



$e = f$

... to get v_Q after Q collides with the wall need to multiply u_Q by e (NEL rearranged):

$\frac{u}{3}(4+10e) \times f$

$= \frac{fu}{3}(4+10e)$

AFTER SECOND COLLISION

the fact that we're talking about a SECOND COLLISION between P and Q implies that we need to find x and compare it with v_Q

↳ hence get ① and ② from part (a)

$$2x + y = 4u \quad ①$$
$$-x + y = 5eu \quad ②$$

and solve for 'x' – elim. 'y'

① – ②

$$3x = 4u - 5eu$$

factorise u and solve for x

Question 3 continued

$$3x = u(4-5e)$$

$\div 3$ $\div 3$

$$x = \frac{u}{3}(4-5e)$$

important to consider now the direction of motion of x - we are given in the question that $e > 4/5$, hence subbing into x gives $x < 0$

↳ hence looks like both particles are moving in the -ve direction - using collision logic, the only way they collide again is when v_B is even more -ve than x (hence catches up with v_A)

↳ formulating the above :

$$-\frac{\cancel{u}f(4+10e)}{\cancel{3}} < \frac{\cancel{u}}{\cancel{3}}(4-5e)$$

v_B $v_A = x$

cancel u's, 3's and $x-1$ (flip inequality sign)

$x-1$ $x-1$

$$f(4+10e) > -(4-5e)$$

expand and $\div(4+10e)$

$$f > \frac{5-4e}{4+10e}$$

however because f is a coefficient of restitution,

$$0 \le e \le 1$$

$\therefore$ full range: $\boxed{\frac{5e-4}{4+10e} < f \le 1}$

Question 3 continued

(Total for Question 3 is 10 marks)

4. A light elastic string has natural length $2a$ and modulus of elasticity $4mg$.
One end of the elastic string is attached to a fixed point O. A particle P of mass m is attached to the other end of the elastic string.
The particle P hangs freely in equilibrium at the point E, which is vertically below O

(a) Find the length OE.

(4)

Particle P is now pulled vertically downwards to the point A, where $OA = 4a$, and released from rest. The resistance to the motion of P is a constant force of magnitude $\frac{1}{4}mg$.

(b) Find, in terms of a and g, the speed of P after it has moved a distance a.

(7)

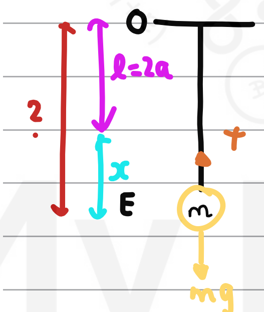
Particle P is now held at O
Particle P is released from rest and reaches its maximum speed at the point B.

The resistance to the motion of P is again a constant force of magnitude $\frac{1}{4}mg$.

(c) Find the distance OB.

(4)

(a) just like with all 'elastic strings and springs' questions, the most important thing is to draw a detailed diagram - the fact that we're given that the particle is in equilibrium means you have to pay particular attention to the FORCES in the string



we can see now that given the 'equilibrium' condition, $\sum F_y = 0$ applies ∴ "forces up = forces down"

$\Rightarrow T = mg$

we can also see from the diagram that

$OE = 2a + x$

hence need to find the string's extension - subbing the information into the strings and springs formula:

formula $T = \frac{\lambda x}{l}$ ← what we need

$\cancel{mg} = \frac{\cancel{4}^{2}\cancel{mg}x}{\cancel{2}a}$ cancel mg's and 2's

and solve for 'x'

$\Rightarrow a = 2x$

$\Rightarrow x = a/2$

hence subbing into our equation for $OE = d$

P 7 2 7 9 8 A 0 1 2 2 8

Question 4 continued

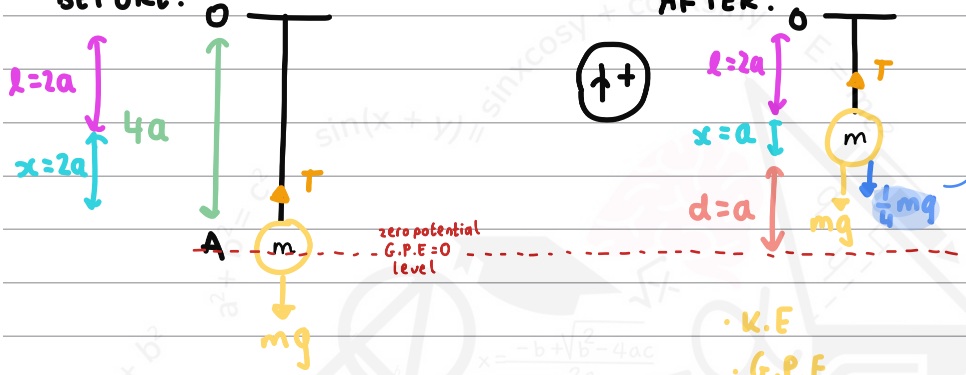
$OE = 2a + a/2$

$\Rightarrow OE = 5/2 a$

(b) the fact that we're asked for the speed of P implies we need to use the energy method of conservation of mechanical energy (including dissipative forces!)

↳ hence let's draw two diagrams: one for before P travels a distance of 'a'm and one for after - labelling the appropriate energies

BEFORE: AFTER:



NOTE: important to label this going opposite to the direction of motion

E.P.E

- K.E
- G.P.E
- w.d by Fr (acts downwards) $= \frac{1}{4}mg$
- E.P.E

now sub all into work-energy principle (includes dissipative forces)

$$\text{w.d in} + K.E_i + G.P.E_i + E.P.E_i = K.E_f + G.P.E_f + E.P.E_f + \text{w.d against friction}$$

w.d in ↓ n/a; $K.E_i$: initial kinetic; $G.P.E_i$: initial gravitational potential; $E.P.E_i$: initial elastic potential; $K.E_f$: final kinetic energy; $G.P.E_f$: final gravitational potential; $E.P.E_f$: final elastic potential

- formula: $\frac{1}{2}mu^2 + mgh_1 + \frac{\lambda x^2}{2l} = \frac{1}{2}mv^2 + mgh_2 + \frac{\lambda x^2}{2l} + Fr \times d$

Subbing into above:

$$0 + 0 + \frac{4\cancel{m}g(2a)^2}{2(2a)} = \frac{1}{2}\cancel{m}v^2 + \cancel{m}ga + \frac{4\cancel{m}g(a)^2}{2(2a)} + \frac{1}{4}\cancel{m}ga$$

cancel m's and expand

$$4ag = \frac{1}{2}v^2 + ag + ag + \frac{1}{4}ag$$

collect like terms

$$\frac{1}{2}v^2 = 7/4 \, ag$$

×2 ×2

solve for v^2

$$v^2 = 7/2 \, ag$$

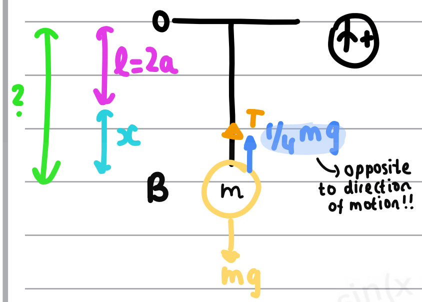
square root

$$\Rightarrow \boxed{v = \pm\sqrt{7/2 \, ag} \ \text{ms}^{-1}}$$

Question 4 continued

(c) METHOD 1: using the equilibrium condition

we know that max speed will be reached when the particle is in equilibrium VERTICALLY $\therefore \sum F_y = 0$ - drawing another diagram to see how we can exploit the equilibrium condition



hence $R(\uparrow)$:

$$T + \frac{1}{4}mg = mg$$

$$\Rightarrow T = \frac{3}{4}mg \quad -①$$

we can also see from the diagram that

$$OB = x + 2a$$

hence need to find this 'x' - subbing into our formula for elastic strings and springs

formula: $T = \frac{\lambda x}{l}$

subbing in ① and the information given

$$\frac{3}{4}m\cancel{g} = \frac{\cancel{4}^{2}m\cancel{g}x}{\cancel{2}a} \leftarrow \text{what we're trying to find}$$

cancel m's and rearrange to solve for 'x'

$$\frac{3}{4} = \frac{2x}{a}$$

$\times a \qquad \times a$

$$2x = \frac{3}{4}a$$

$\div 2 \qquad \div 2$

$$x = \frac{3}{8}a$$

subbing into our equtn for OB:

$$OB = \frac{3}{8}a + 2a$$

$$\boxed{= \frac{19}{8}a \text{ m}}$$

METHOD 2: using principle of conservation of mechanical energy and differentiation

key here is to draw two diagrams - one for when P is held 'at rest' at O and second when reaches max speed - call it $v = V$ when spring has extended 'x'm to point B where $OB = $ 'h' - label appropriate energies

Que now sub all into work-energy principle (includes dissipative forces)

w.d in + $K.E_i$ + $G.P.E_i$ + $E.P.E_i$ = $K.E_f$ + $G.P.E_f$ + $E.P.E_f$ + w.d against friction

w.d in: n/a; $K.E_i$: initial kinetic; $G.P.E_i$: initial gravitational potential; $E.P.E_i$: initial elastic potential; $K.E_f$: final kinetic energy; $G.P.E_f$: final gravitational potential; $E.P.E_f$: final elastic potential

formula: $\frac{1}{2}mu^2 + mgh_1 + \frac{\lambda x^2}{2l} = \frac{1}{2}mv^2 + mgh_2 + \frac{\lambda x^2}{2l} + Fr \times d$

Subbing into above: *solve for 'h'

$$0 + \cancel{m}gh + \frac{4\cancel{m}g(2a)^2}{2(2a)} = \frac{1}{2}\cancel{m}v^2 + \frac{4\cancel{m}g(h-2a)^2}{2(2a)} + \frac{1}{4}\cancel{m}gh$$

cancel m's and expand

$$gh + 4ag = \frac{1}{2}v^2 + \frac{g(h-2a)^2}{a} + \frac{1}{4}gh$$

rearrange - make $\frac{1}{2}v^2$ the subject

$$\frac{1}{2}v^2 = gh + 4ag - \frac{g(h-2a)^2}{a} - \frac{1}{4}gh$$

collect like terms

$$\frac{1}{2}v^2 = \frac{3}{4}gh + 4ag - \frac{g(h-2a)^2}{a}$$

×2 ×2

$$v^2 = \frac{3}{2}gh + 8ag - \frac{2g(h-2a)^2}{a}$$

exploiting fact that $\frac{dv^2}{dh} = 0$

$$\left(\frac{dv^2}{dh} = 0\right) \Rightarrow \frac{3}{2}g - \frac{4g(h-2a)}{a} = 0$$

solve for 'h'

$$\frac{3}{2}\cancel{g} = \frac{4\cancel{g}(h-2a)}{a} \quad \text{cancel } g\text{'s}$$

$$\frac{3}{2}a = 4h - 8a$$

$$\Rightarrow 4h = \frac{19}{2}a \quad \div 4$$

$$\Rightarrow h = \frac{19}{8}a$$

$$\boxed{\therefore OB = 19/8\, a\, m}$$

(Total for Question 4 is 15 marks)

5.

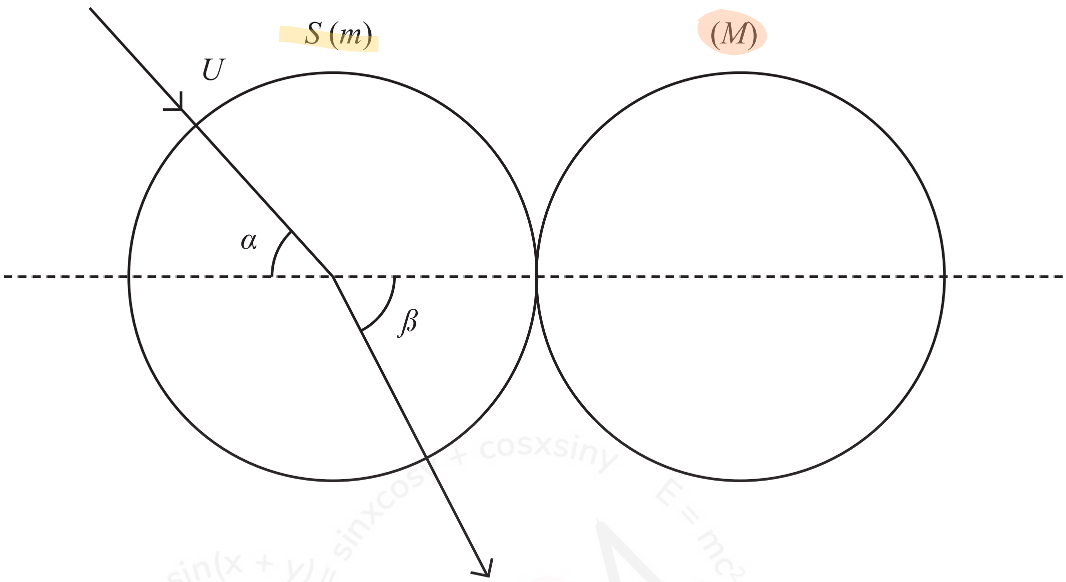


Figure 1

A smooth uniform sphere S of mass m is moving with speed U on a smooth horizontal plane. The sphere S collides obliquely with another uniform sphere of mass M which is at rest on the plane. The two spheres have the same radius.

Immediately before the collision the direction of motion of S makes an angle α, where $0 < \alpha < 90°$, with the line joining the centres of the spheres.

Immediately after the collision the direction of motion of S makes an angle β with the line joining the centres of the spheres, as shown in Figure 1.

The coefficient of restitution between the spheres is e.

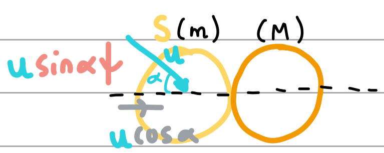
(a) Show that $\tan\beta = \dfrac{(m+M)\tan\alpha}{(m-eM)}$ **(8)**

Given that $m = eM$,

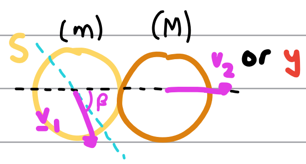
(b) show that the directions of motion of the two spheres immediately after the collision are perpendicular. **(2)**

(a) notice we have an 'oblique collisions between two spheres' question - splitting the diagram in Fig 1 into before and after

BEFORE



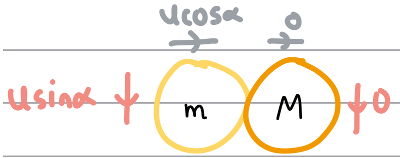
AFTER:



...resolve into i-j components:

...resolve into i-j components:

Question 5 continued



↳ for perp. comps no change as Impulse only acts parallel to the line of centres

because we want an expression for $\tan\beta$, we need to find the perpendicular and parallel components of v_1

... perpendicular:

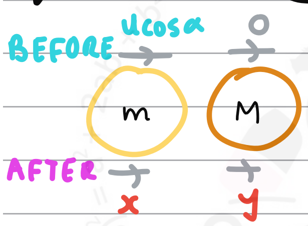
as seen - no change

$\Rightarrow v_1\sin\beta = U\sin\alpha$

... parallel:

WAY 1: label parallel comps as 'x' and 'y'

treat as standard elastic collisions in 2D question



• using PCLM for 'x'. means total momentum before equals total momentum after

formula: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

sub into above

$mU\cos\alpha = mx + My$

$\Rightarrow mx + My = mU\cos\alpha$ — ①

• next, NEL (Impact law):

formula: $e = \dfrac{\text{speed of separation}}{\text{speed of approach}} = \dfrac{y - x}{U\cos\alpha}$

$\times U\cos\alpha$ $\quad\quad \times U\cos\alpha$

$y - x = eU\cos\alpha$ — ②

solve ① and ② simultaneously - elim. 'y'

① − ② × M

$mx + My = mU\cos\alpha$

$-\;-Mx + My = MeU\cos\alpha$

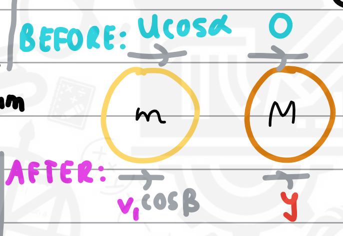
$(m+M)x = mU\cos\alpha - MeU\cos\alpha$

factorise $U\cos\alpha$ out

$(m+M)x = U\cos\alpha(m - eM)$

$\div(m+M) \quad \div(m+M)$

WAY 2: labelling the parallel comps as '$v_1\cos\beta$' and 'y'



• first, using PCLM - means total momentum before equals total momentum after

formula: $m_A u_A + m_B u_B = m_A u_A + m_B u_B$

sub into above

$U\cos\alpha\, m = v_1\cos\beta\, m + My$

$\Rightarrow v_1 m\cos\beta + My = Um\cos\alpha$ — ①

• next, NEL (Impact law):

formula: $e = \dfrac{\text{speed of separation}}{\text{speed of approach}} = \dfrac{y - v_1\cos\beta}{U\cos\alpha}$

$\Rightarrow -v_1\cos\beta + y = eU\cos\alpha$ — ②

AND know ... parallel:

no impact ∴ $v_1\sin\beta = U\sin\alpha$

$\Rightarrow U = \dfrac{v_1\sin\beta}{\sin\alpha}$ — ③

subbing ② and ③ into ①

$v_1 m\cos\beta + M(v_1\cos\beta + eU\cos\alpha) = \dfrac{v_1\sin\beta}{\sin\alpha}\, m\cos\alpha$

sub in ③ again and simplify: $\dfrac{\cos\alpha}{\sin\alpha} = \cot\alpha$

Question 5 continued

$x = \frac{u\cos\alpha(m-eM)}{m+M}$

∴ looking at our first 'after' diagram, we can infer that:

$\tan\beta = \frac{\text{perp.}}{\text{parallel}}$

$\tan\beta = \frac{u\sin\alpha}{\frac{u\cos\alpha(m-eM)}{m+M}}$

and now just the case of trying to make it look like the equation given in the question:

$\times(m+M)$ $\times(m+M)$

$\tan\beta = \frac{(m+M)\cancel{u}\sin\alpha}{(m-eM)\cancel{u}\cos\alpha}$

cancel u's and use $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$\Rightarrow \boxed{\tan\beta = \frac{(m+M)\tan\alpha}{m-eM}}$

as required

$m\cos\beta + M(\cos\beta + e\sin\beta\cot\alpha) = m\sin\beta\cot\alpha$

need to try make 'tanα' and 'tanβ'

$\div\cos\beta$ $\div\cos\beta$

$m + M(1 + e\tan\beta\cot\alpha) = m\tan\beta\cot\alpha$

expand brackets

$m + M + Me\tan\beta\cot\alpha = m\tan\beta\cot\alpha$

collect 'like' expressions

$m + M = m\tan\beta\cot\alpha - Me\tan\beta\cot\alpha$

factorise $m - M$

$m + M = (m - eM)\tan\beta\cot\alpha$

$\div\cot\alpha$ $\div\cot\alpha$

$(m+M)\tan\alpha = (m-eM)\tan\beta$

$\div m-eM$ $\div m-eM$

$\boxed{\tan\beta = \frac{(m+M)\tan\alpha}{m-eM}}$

as required

(b) to show that the direction of motion of the two spheres after the collision are perpendicular, must show that the x / $v_1\cos\beta = 0$ OR that $\beta = 90°$

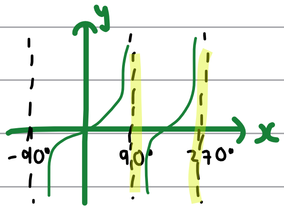
...considering the latter:

know from (a) that $\tan\beta = \frac{(m+M)\tan\alpha}{m-eM}$

now if $m = eM$,

dividing by 0 gives ∞, so

looking at tan graph to see where $\tan\beta = \infty$



$\boxed{\therefore \beta = 90°}$

↳ in this case the diagram for after the collision would be

∴ moving perp. to each other

P72798A01828

Question 5 continued

(Total for Question 5 is 10 marks)

6. A particle P of mass m is falling vertically when it strikes a fixed smooth inclined plane. The plane is inclined to the horizontal at an angle α, where $0 < \alpha \leqslant 45°$

At the instant immediately before the impact, the speed of P is u.

At the instant immediately after the impact, P is moving horizontally with speed v.

(a) Show that the magnitude of the impulse exerted on the plane by P is $mu\sec\alpha$

(5)

The coefficient of restitution between P and the plane is e, where $e > 0$

(b) Show that $v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$

(3)

(c) Show that the kinetic energy lost by P in the impact is

$$\frac{1}{2}mu^2(1 - e^2)\cos^2\alpha$$

(2)

(d) Hence find, in terms of m, u and e **only**, the kinetic energy lost by P in the impact.

(2)

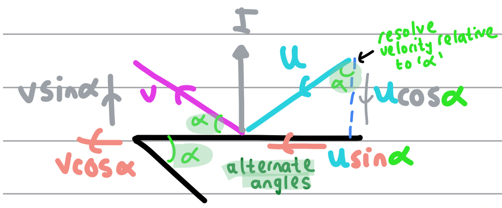
(a) illustrating this oblique collision between the particle and the inclined plane diagrammatically - make sure to clearly label the direction of motion, the respective speeds etc.

↳ NOTE: always easier for 'falling vertically' questions to turn the axis so that the inclined plane looks like it's a horizontal surface

3 WAYS TO DO THIS QUESTION:

METHOD 1: getting expression for e

↳ NOTE: even though this is the longest out of the three ways to do part (a), it's necessary to then work out the other parts of the question faster - see that part (a) is worth more marks than the other parts so seems reasonable to consider this Method 1 as the best method



... perp:

IMPULSE acts perp. to the fixed surface ∴ NEL rearranged applies

⇒ $v\sin\alpha = eu\cos\alpha$

... parallel:

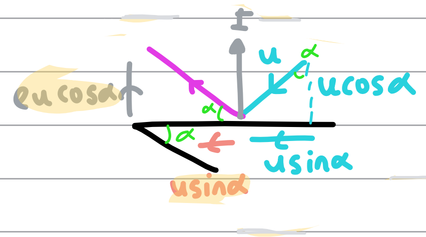
↳ these do not change as no impulse

⇒ $v\cos\alpha = u\sin\alpha$

www.mymathscloud.com

Question 6 continued

...populating this onto the diagram:



from this,

$\tan\alpha = \dfrac{e\cancel{u}\cos\alpha}{\cancel{u}\sin\alpha}$

$\Rightarrow \tan\alpha = \dfrac{e\cos\alpha}{\sin\alpha}$

simplifying $\dfrac{\cos\alpha}{\sin\alpha} = \cot\alpha$

$\tan\alpha = e\cot\alpha$

$\times\tan\alpha \quad \times\tan\alpha$

$\Rightarrow e = \tan^2\alpha$

...applying Impulse-momentum formula perp. to the plane:

formula: $I = m(v - u)$

$I = m(eu\cos\alpha - (-u\cos\alpha))$

$\Rightarrow I = m(eu\cos\alpha + u\cos\alpha)$

Sub in our $e = \tan^2\alpha$ result

$I = m(\tan^2\alpha\, u\cos\alpha + u\cos\alpha)$

$= mu\left(\dfrac{\sin^2\alpha}{\cancel{\cos^2\alpha}}\cancel{\cos\alpha} + \cos\alpha\right)$ $\cos\alpha$

$\cos\alpha \div \frac{1}{\cos\alpha} = \cos^2\alpha$

factorise $1/\cos\alpha$ out to try get identity we recognise:

$\Rightarrow I = \dfrac{mu}{\cos\alpha}(\sin^2\alpha + \cos^2\alpha)$

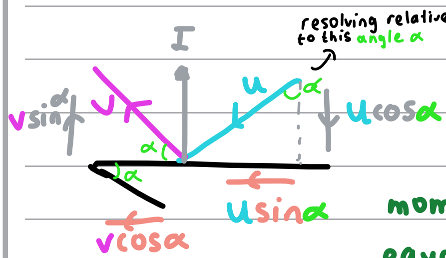
$\Rightarrow I = \dfrac{mu}{\cos\alpha}(1)$

and using $\dfrac{1}{\cos\alpha} = \sec\alpha$

to finally get: $\boxed{I = mu\sec\alpha}$ as required

WAY 2: PCLM parallel and Impulse-momentum perpendicular

...parallel comps:



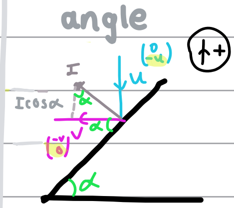
, we can use PCLM parallel to plane - means total momentum before equals total momentum after (as no impulse)

formula: $m_Au_A + m_Bu_B = m_Av_A + m_Bv_B$

WAY 3: using VECTORS

METHOD 1: only vertical comps

for this way we're not turning the axis as important to get the Impulse angle



subbing into Impulse-momentum formula (relative to 'upward' direction)

formula: $I = m(\underline{v} - \underline{u})$

$I\cos\alpha = m(0 - (-u))$

$I\cos\alpha = mu$

P72798A02128

Question 6 continued

Sub into above:

$\cancel{m}(u\sin\alpha)=\cancel{m}(v\cos\alpha)$

cancel m's and solve for 'v'

$\Rightarrow v=\dfrac{u\sin\alpha}{v\cos\alpha}$ – ①

...perp. components:

here, sub into the impulse-momentum formula:

formula: $I=m(\underline{v}-\underline{u})$

$I=m\left(v\sin\alpha-(-u\cos\alpha)\right)$

$\Rightarrow I=m(v\sin\alpha+u\cos\alpha)$

sub ① into above:

$I=m\left(\dfrac{u\sin\alpha}{\cos\alpha}(\sin\alpha)+u\cos\alpha\right)$

expand

$u\cos\alpha\div\dfrac{u}{\cos\alpha}=u\cos\alpha\times\dfrac{\cos\alpha}{u}=\cos^2\alpha$

$I=m\left(\dfrac{u\sin^2\alpha}{\cos\alpha}+u\cos\alpha\right)$

factorise $\dfrac{u}{\cos\alpha}$ out to try get identity we recognise:

$I=\dfrac{mu}{\cos\alpha}(\sin^2\alpha+\cos^2\alpha)$

using identity: $\sin^2\alpha+\cos^2\alpha=1$

$\therefore I=\dfrac{mu}{\cos\alpha}$ and know $\dfrac{1}{\cos\alpha}=\sec\alpha$

$\Rightarrow \boxed{I=mu\sec\alpha}$ as required

and rearranging for I:

$\div\cos\alpha$ $\div\cos\alpha$

$\therefore I=\dfrac{mu}{\cos\alpha}$ but $\dfrac{1}{\cos\alpha}=\sec\alpha$

$\therefore \boxed{I=mu\sec\alpha}$

METHOD 2: using vector impulse momentum

... parallel:

using PCLM (as in WAY 2):

$\cancel{m}(u\sin\alpha)=\cancel{m}(v\cos\alpha)$

$\Rightarrow v=\dfrac{u\sin\alpha}{\cos\alpha}=u\tan\alpha$ – ①

... perp:

using vector impulse-momentum:

formula: $I=m(\underline{v}-\underline{u})$

$=m\left(\begin{pmatrix}-v\\0\end{pmatrix}-\begin{pmatrix}0\\-u\end{pmatrix}\right)$

$I=m\begin{pmatrix}-v\\u\end{pmatrix}$

but need the magnitude ∴ Pythag.

$|I|=m\sqrt{(-v)^2+(u)^2}$

subbing in ① into above

$|I|=m\sqrt{(-u\tan\alpha)^2+u^2}$

$=m\sqrt{u^2\tan^2\alpha+u^2}$

factorise 'u' out:

$|I|=mu\sqrt{\tan^2\alpha+1}$

use $\sec^2\alpha=1+\tan^2\alpha$

$\Rightarrow \boxed{|I|=mu\sec\alpha}$ as required

(b) ...ctd. from explanations in METHOD 1:

see that $v=\begin{pmatrix}-u\sin\alpha\\eu\cos\alpha\end{pmatrix}$

∴ squaring v's i and j comps to get the 'v^2' in the question

$v^2=(-u\sin\alpha)^2+(eu\cos\alpha)^2$

$=u^2\sin^2\alpha+e^2u^2\cos^2\alpha$

factorise u^2

$\boxed{v^2=u^2(\sin^2\alpha+e^2\cos^2\alpha)}$ as required

Question 6 continued

(c) to get the K.E lost, sub into the formula for E_k:

formula: $E_{k(lost)} = \frac{1}{2}m(u^2 - v^2)$

↳ from part (b)

$= \frac{1}{2}m\left(u^2 - \left(u^2(\sin^2\alpha + e^2\cos^2\alpha)\right)\right)$

factorise u^2 out:

$= \frac{1}{2}mu^2\left(1 - \sin^2\alpha - e^2\cos^2\alpha\right)$

now just the case of making the above expression look like that in the question

↳ get rid of $\sin^2\alpha$ by subbing in our identity rearranged:

$\sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = 1 - \sin^2\alpha$

$= \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$

factorise $\cos^2\alpha$ out

E_k lost $= \frac{1}{2}mu^2\cos^2\alpha(1 - e^2)$

or $\boxed{\frac{1}{2}mu^2(1-e^2)\cos^2\alpha}$ as required

(d) if want to get the expression for E_k lost in terms of m, u and e only, then need to find a way to get rid of the $\cos^2\alpha$

↳ we know from part (a)'s Method 1 that $e = \tan^2\alpha$ ①

manipulating this to try get $\cos^2\alpha$

↳ let's use the identity: $\sec^2\alpha = 1 + \tan^2\alpha$

$\Rightarrow \tan^2\alpha = \sec^2\alpha - 1$

sub in ①

$e = \sec^2\alpha - 1$

using $\sec^2\alpha = \frac{1}{\cos^2\alpha}$

$e = \frac{1}{\cos^2\alpha} - 1$

$\therefore \frac{1}{\cos^2\alpha} = e + 1$

reciprocal

$\Rightarrow \cos^2\alpha = \frac{1}{e+1}$

subbing this into part c's answer to get:

$\boxed{E_k \text{ lost} = \frac{1}{2}mu^2\left(\frac{1}{e+1}\right)(1-e^2)}$

(Total for Question 6 is 12 marks)

7.

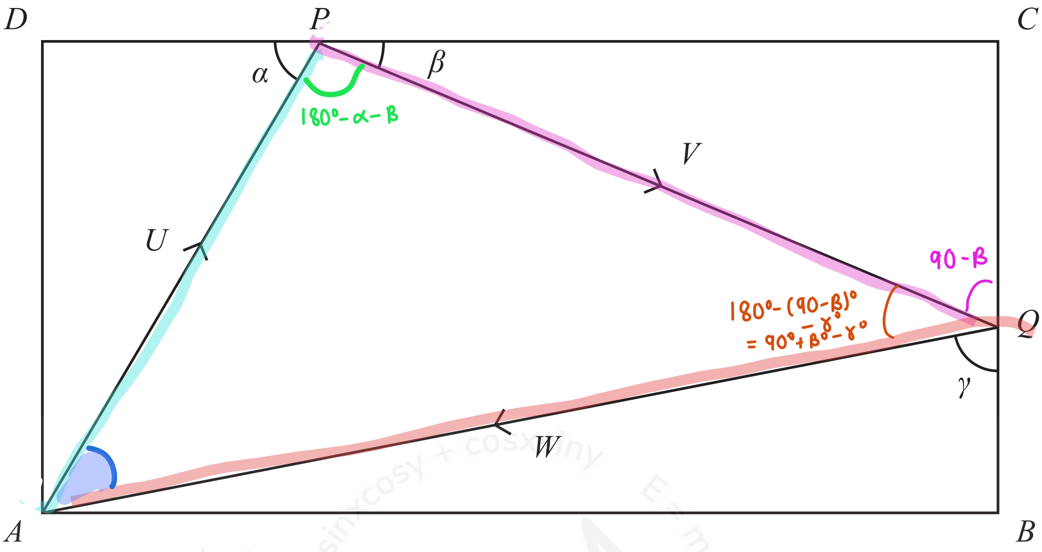


Figure 2

A small smooth snooker ball is projected from the corner A of a horizontal rectangular snooker table $ABCD$.

The ball is projected so it first hits the side DC at the point P, then hits the side CB at the point Q and then returns to A.

Angle $APD = \alpha$, Angle $QPC = \beta$, Angle $AQB = \gamma$
The ball moves along AP with speed U, along PQ with speed V and along QA with speed W, as shown in Figure 2.

The coefficient of restitution between the ball and side DC is e_1

The coefficient of restitution between the ball and side CB is e_2

The ball is modelled as a particle.

Use the model to answer all parts of this question.

(a) Show that $\tan\beta = e_1 \tan\alpha$ **(4)**

(b) Hence show that $e_1 \tan\alpha = e_2 \cot\gamma$ **(3)**

(c) By considering (angle APQ + angle AQP) or otherwise, show that it would be possible for the ball to return to A only if $e_2 > e_1$ **(6)**

If instead $e_1 = e_2$, the ball would **not** return to A.

Given that $e_1 = e_2$

(d) use the result from part (b) to describe the path of the ball after it hits CB at Q, explaining your answer. **(1)**

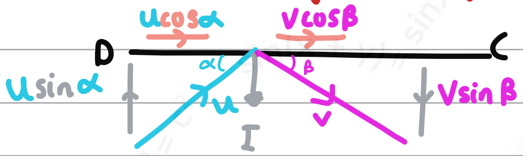
Question 7 continued

realising this is a successive oblique collisions question - probably the most important thing to do first is to label the angles that each of the velocities make with their respective sides (exploiting basic 'angle in a triangle' rules) - in this way we can get FIXED ANGLES to which we can relate our velocities as the question proceeds

↳ see Figure 2

(a) the fact that we're asked to find $\tan\beta$ implies we need the perpendicular and parallel components of the velocity after the first collision - i.e the one between the ball and the side DC

↳ illustrating this diagrammatically:



... parallel components:

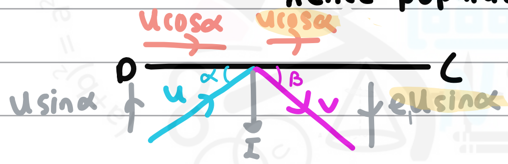
here, no impulse acts parallel to the line of centres ∴ no change in the velocity components

$\Rightarrow v\cos\beta = u\cos\alpha$

... perpendicular components:

Impulse acts perp. to the line of centres ∴ NEL rearranged applies $\Rightarrow v\sin\beta = e_1 u\sin\alpha$

hence populating this onto our diagram:



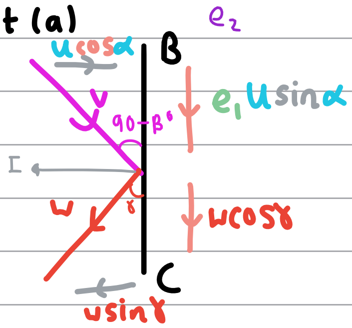
∴ from this, let's derive the expression for

$\tan\beta = \frac{O}{H} = \frac{e_1 \cancel{u}\sin\alpha}{\cancel{u}\cos\alpha}$ cancel u's and use $\frac{\sin\alpha}{\cos\alpha} \equiv \tan\alpha$

$\boxed{\therefore \tan\beta = e_1\tan\alpha}$ as required

(b) see on the RHS of the given equation that the angle 'γ' is included - this suggests we need to find the parallel and perpendicular components of the velocity after the second collision - the one between the ball and the side BC

↳ illustrating this diagrammatically (and using our velocity after components from part (a)



... parallel component:

↳ no impulse ∴ no change

$\Rightarrow w\cos\gamma = e_1 u\sin\alpha$

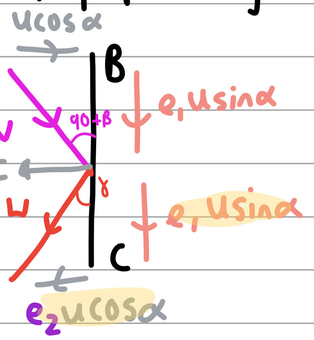
... perpendicular component:

↳ here, Impulse acts ∴ NEL rearranged applies

$\Rightarrow w\sin\gamma = e_2 u\cos\alpha$

Question 7 continued

...populating this onto our diagram:



from this, similarly as in part (a) can derive formula for $\tan\gamma = \frac{O}{H} = \frac{e_2 u\cos\alpha}{e_1 u\sin\alpha}$

cancel u's

$$\Rightarrow \tan\gamma = \frac{e_2\cos\alpha}{e_1\sin\alpha}$$

but because question includes $\cot\gamma$ not $\tan\gamma$, need to find reciprocal of above (flip numerator and denominator)

$$\Rightarrow \cot\gamma = \frac{e_1\sin\alpha}{e_2\cos\alpha}$$

using $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$$\Rightarrow \cot\gamma = \frac{e_1}{e_2}\tan\alpha$$

$\times e_2 \qquad \times e_2$

$$e_2\cot\gamma = e_1\tan\alpha$$

or $\boxed{e_1\tan\alpha = e_2\cot\gamma}$ as required

(c) WAY 1: using angle $P\hat{A}Q$ and part (b)

from angles in a triangle

the fact that we're asked to consider $A\hat{P}Q + A\hat{Q}P$ implies we should be looking at finding $P\hat{A}Q = 180° - (A\hat{P}Q + A\hat{Q}P)$

$= 180° - (180° - \alpha° - \beta°) - (90° + \beta° - \gamma°)$

$= \alpha° + \gamma° - 90°$

↳ now, logically the only way it would be possible for the ball to return to A is

if $P\hat{A}Q > 0$, or

$\alpha° + \gamma° - 90° > 0$

$\Rightarrow \alpha > 90° - \gamma°$

noticing a similar relationship between the two angles in (b) ∴ taking tan of both sides

$\tan\alpha > \tan(90° - \gamma°)$

WAY 2: using angles $A\hat{P}Q$ and $A\hat{Q}P$ and trig addition rule

alternatively, can exploit the fact that $P\hat{A}Q$, $A\hat{P}Q$ and $A\hat{Q}P$ all form a triangle, so $A\hat{P}Q + A\hat{Q}P < 180°$

$\Rightarrow 180° - \alpha° - \beta° + 90° + \beta° - \gamma° < 180°$

$270° - \alpha° - \gamma° < 180°$

$\therefore \alpha° > 90° - \gamma°$

see similar expression in part (b) ∴ let's take tan of both sides

$\tan\alpha > \tan(90° - \gamma°)$

using $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$ then addition rule

formula: $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\sin B \mp \sin A\cos B$

$$\tan\alpha > \frac{\sin(90° - \gamma°)}{\cos(90° - \gamma°)}$$

Question 7 continued

but know complementary angle rule that $\tan(90°-\gamma)=\cot\gamma$

$\therefore \tan\alpha > \cot\gamma$

↳ subbing in terms from (b) rearranged

$\frac{e_2\cancel{\cot\gamma}}{e_1} > \cancel{\cot\gamma}$

$\times e_1 \quad \times e_1$

$\boxed{e_2 > e_1}$

as required

$\Rightarrow \tan\alpha > \frac{\sin 90°\cos\gamma - \cos 90°\sin\gamma}{\cos 90°\cos\gamma + \sin 90°\sin\gamma}$

$\sin 90° = 1,\ \cos 90° = 0$

$\Rightarrow \tan\alpha > \frac{\cos\gamma}{\sin\gamma}$ identity: $\frac{\cos\alpha}{\sin\alpha} \equiv \cot\alpha$

$\Rightarrow \tan\alpha > \cot\gamma$

subbing in part (b)'s answer rearranged

$\frac{e_2\cancel{\cot\gamma}}{e_1} > \cancel{\cot\gamma}$

$\times e_1 \quad \times e_1$

$\Rightarrow \boxed{e_2 > e_1}$

as required

(d) from (b): $e_1\tan\alpha = e_2\cot\gamma$

but if $e_1 = e_2$ (as given in question), then

$\Rightarrow \tan\alpha = \cot\gamma$

$\therefore \alpha = 90° - \gamma$ (from tan and cot's complementary angle relationship)

$\boxed{\therefore P \text{ would move parallel to the initial velocity}}$

Question 7 continued

(Total for Question 7 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS